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# SOME THEOREMS CONCERNING GROUPS WHOSE ORDERS ARE POWERS OF A PRIME \*

BY

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Professor Burnside has called attention to certain limitations on the nature of the derived groups of groups of prime power order.† One of these limitations is that such a derived group cannot be non-abelian and have a cyclic central. Theorem I of the present paper gives a slight generalization of this limitation, and an obvious modification of the argument used in the proof of this theorem establishes the important result that the second central of a group whose order is a power of an odd prime cannot be cyclic.

The results given at the end of the paper are continuations of some obtained by Professor Burnside in the article just cited.

Throughout the paper frequent use is made of the theorem that in a group whose order is a power of a prime the operations which correspond to the invariant operations of the  $i$ th cogredient are commutative with all of the  $i$ th commutators of the group.‡

We shall represent by  $G$  a group of class  $k$  and order  $p^m$ , where  $p$  is a prime; by  $G'$  the first cogredient of  $G$ ; by  $H_i$  the  $i$ th central§ of  $G$ ; by  $K_j$  the  $j$ th commutator subgroup; and by  $K_{j,i}$  the subgroup of  $K_j$  that is contained in  $H_i$ .

Suppose that  $K_{j,2}$  is cyclic and generated by the operation  $t$  of order  $p^a$ . If  $k > 3$ , every operation of  $K_{j,3}$  of order  $p$  must be contained in  $K_{j,2}$  since the operations of this latter group are invariant in  $K_j$  and its operations of order  $p$  are invariant in  $G$ . Hence  $K_{j,3}$  has only one subgroup of order  $p$ . It follows at once that  $K_{j,3}$  is cyclic if  $p$  is odd. If  $p = 2$ ,  $K_{j,2}$  contains an operation of order  $2^2$  that is invariant in  $K_{j,3}$ . But in the non-cyclic group of order a power of 2 with only one subgroup of order 2 no operation of

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† Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912), pp. 241, 242.

‡ Fite, these Transactions, vol. 7 (1906), p. 62.

§ I follow here the notation of Burnside's *Theory of Groups*, first edition, p. 62, rather than that of the second edition, p. 120. Cf. de Séguier, *Éléments de la théorie des groupes abstraits*, p. 87.

order  $2^2$  is invariant. We conclude therefore that in this case also  $K_{j, 3}$  is cyclic.

Suppose then that  $K_{j, 3}$  is generated by the operation  $t_1$  of order  $p^{a+\beta}$ . If we denote by  $K'_{j, i}$  the subgroup of the  $j$ th commutator subgroup of  $G'$  that is contained in the  $(i - 1)$ th central of  $G'$ , it follows from what has been proved that  $K'_{j, 4}$  must be cyclic. Hence the quotient group  $K_{j, 4} / K_{j, 3}$  must be cyclic. If it is of order  $p^\gamma$  and  $t_2$  is an operation of  $K_{j, 4}$  that corresponds to a generator of it,  $t_2^\gamma$  is a power of  $t_1$  whose exponent is relatively prime to  $p$ , and therefore  $K_{j, 4}$  is cyclic. An obvious continuation of this argument shows that  $K_j$  is cyclic. This establishes the following

**Theorem I.** *The  $j$ th commutator subgroup of a group of prime power order is cyclic if those of its operations that are contained in the second central of the group form a cyclic subgroup.*

Since  $K_{1, 2}$  is contained in the central of  $K_1$ , Theorem I includes as a special case Professor Burnside's theorem referred to in the introduction.

In case  $p$  is odd an argument closely analogous to the one used in the proof of Theorem I shows that if  $H_2$  were cyclic  $G$  itself would be cyclic. But if  $G$  were cyclic there would be no  $H_2$ . This proves

**Theorem II.** *The second central of a group of order  $p^m$ , where  $p$  is an odd prime, cannot be cyclic.*

If  $p = 2$ , the preceding argument does not apply, since the operations of  $H_2$  are not all necessarily invariant in  $H_3$  and therefore an operation of  $H_3$  of order 2 may give a commutator of order  $2^2$ . We can however proceed as follows.

If  $k > 3$ ,  $H_3$  contains a commutator  $t_1$  that is not in  $H_2$ . Now  $\{H_2, t_1\}$  is an abelian group with only one subgroup of order 2, since  $t_1$  is commutative with every one of its commutators. Hence this group is cyclic. If  $H_3$  contained an operation  $s$  of order 2 that is not in  $\{H_2, t_1\}$ ,  $s$  would transform  $t_1$  into itself multiplied by an invariant commutator of  $G$  of order 2. Hence  $s$  would be commutative with  $t_1$ , and accordingly every operation of  $G$  would transform it into itself multiplied by an invariant commutator of  $G$ . But this is not consistent with the fact that  $s$  is not in  $H_2$ . We conclude therefore that  $H_3$  contains only one subgroup of order 2. It contains the operation  $t_1$  and every one of its operations transforms  $t_1$  into itself multiplied by an operation of  $H_1$ . But this would be impossible in case  $H_3$  were non-cyclic. We conclude therefore that  $H_3$  is cyclic.

The continuation of the argument up to the point of showing that  $H_{k-1}$  must be cyclic is the same as in the case of an odd prime. But beyond this point the argument breaks down, since  $G / H_{k-1}$  has no commutator besides the identity. As a matter of fact, there are groups of order  $2^m$  whose second centrals are cyclic—for example, the dihedral groups of these orders when  $m > 3$ .

Suppose now that  $K_1$  is cyclic and generated by the commutator  $t$  of order

$p^\lambda$ , where  $p$  is odd. If  $k > 3$ ,  $G/H_{k-2}$  is metabelian\* with a cyclic commutator subgroup. If  $p^r$  is the order of this subgroup,  $G/H_{k-3}$  must contain an invariant commutator of order  $p^r$  and the invariant commutators of  $G/H_{k-3}$  must form a cyclic subgroup of order  $p^s$ , where  $s \geq r$ .

Since any operation of  $G/H_{k-3}$  that corresponds to an invariant operation of  $G/H_{k-2}$  is commutative with every commutator of  $G/H_{k-3}$ , no operation of  $H_{k-1}$  can transform  $t$  into itself multiplied by a commutator that is not in  $H_{k-3}$ . But some operation of  $G$  must transform  $t$  into  $t^{1+xp^{s_1}}$ , where  $x$  is relatively prime to  $p$ , and  $r \leq s_1 < r+s$ , since otherwise  $t$  would be contained in  $H_{k-2}$  and  $G/H_{k-2}$  would be abelian. But this is impossible since  $G$  is of class  $k$ .

Suppose then that  $A$  is so selected that

$$A^{-1}tA = t^{1+xp^{s_1}},$$

where  $x$  is relatively prime to  $p$ , and  $s_1$  is a minimum. Then

$$A^{-1}t^{p^{r-1}}A = t^{p^{r-1}+xp^{r+s_1-1}}, \quad A^{-1}t^{p^r}A = t^{p^r+xp^{r+s_1}}.$$

Now since  $t^{p^r}$  is contained in  $H_{k-2}$ , while  $t^{p^{r-1}}$  is not, it follows that

$$r+s_1 \geq r+s, \quad r+s_1-1 < r+s.$$

Hence  $s_1 = s$ , and therefore

$$A^{-p^r}tA^{p^r} = t^{(1+xp^s)p^r}.$$

But

$$(1+xp^s)^{p^r} \equiv 1+xp^{r+s} \pmod{p^{2s+1}}.$$

Moreover  $r+s < 2s+1 \leq \lambda$ , since the lowest power of  $t$  that is contained in  $H_{k-3}$  is the  $p^{r+s}$ th power and since  $H_{k-3}$  must contain a commutator of order  $p^s$ . Hence

$$(1+xp^s)^{p^r} \not\equiv 1 \pmod{p^\lambda},$$

and  $A^{p^r}$  is not commutative with  $t$ . Since however  $G/H_{k-2}$  is metabelian and contains no commutator of order greater than  $p^r$ ,  $A^{p^r}$  must be contained in  $H_{k-1}$ . We conclude therefore that if  $K_1$  is cyclic and  $k > 3$ , not all of the operations of  $K_1$  are invariant in  $H_{k-1}$ .

This argument with a slight modification is valid when  $p = 2$ , provided that  $s > 1$ ; but when  $s = 1$  it breaks down and the conclusion does not hold, as may be seen from the groups of order  $2^5$ .

We have assumed that  $k > 3$ . The existence of metabelian groups with cyclic commutator subgroups shows that the conclusion does not hold when

\* I use the term *metabelian* in the sense defined by me in these Transactions, vol. 3 (1902), p. 331. Burnside uses the term in a different sense. See his *Theory of Groups*, second edition, p. 57.

$k = 2$ . When  $k = 3$  every operation of  $H_2$  is commutative with  $t$ . But the following conclusion from this argument for  $k > 3$  does hold when  $k = 3$ ; namely,\* if  $K_1$  is cyclic,  $H_{k-1}$  cannot coincide with  $K_1$ .

The commutator formed by any two operations,  $t_1$  and  $t_2$ , of  $K_1$  is contained in  $H_{k-3}$ , since  $G/H_{k-3}$  is of class three and the commutator subgroup of a group of the third class is abelian. The index of  $K_1$  under  $K_1$  is at least  $p^2$ . If it is exactly  $p^2$ , we can assume that  $t_2$  is equal to the product of some operation of  $H_{k-2}$  and  $t_1$ . Hence† the commutator formed by  $t_1$  and  $t_2$  is in  $H_{k-4}$  and the index of the commutator subgroup of  $K_1$  under  $K_1$  is at least  $p^3$ .

If  $t_1$  and  $t_2$  are operations of  $K_i$  they are  $i$ th commutators or products of  $i$ th commutators and correspond to invariant operations of the  $i$ th cogredient of  $G/H_{k-2i-1}$ , since they are contained in  $H_{k-i}$ . Hence the commutator formed by  $t_1$  and  $t_2$  is contained in  $H_{k-2i-1}$ . Now if  $K_i$  is non-abelian the index of  $K_i$  under  $K_i$  is at least  $p^{i+1}$ . If it is exactly  $p^{i+1}$ , we can show by an argument similar to the one used in the special case just considered that the commutator formed by  $t_1$  and  $t_2$  is contained in  $H_{k-2i-2}$ . Hence the index of the commutator subgroup of  $K_i$  under  $K_i$  is at least  $p^{i+2}$  if  $K_i$  is non-abelian.

If the  $j$ th derived group is contained in  $H_x$ , the  $(j+1)$ th derived group is contained in  $H_{x-(j+1)}$ , provided that  $x > j+1$ . Hence the  $i$ th derived group is contained in  $H_{k-i(i+1)/2}$ , and therefore in a group of order  $p^m$  whose  $i$ th derived group is not the identity we must have  $k > i(i+1)/2$ . Hence  $m \geq 2 + i(i+1)/2$ . For  $i > 4$  this exceeds the lower limit for  $m$  given by Professor Burnside.

COLUMBIA UNIVERSITY,  
July, 1913.

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\* de Séguier, *loc. cit.*, p. 127. This follows also from Theorem II.

† This result is given by Burnside in the article cited in the introduction.